Announcement: HW2 posted on website, due 8/10 2359

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Department of Mathematics

MATH2058 Honours Mathematical Analysis I
Tutorial 3
Date: 27 September, 2024

Field Axioms of real number:

A1. $a+b \in \mathbb{R}$ if $a,b \in \mathbb{R}$;

A2. $a+b = b+a$ if $a,b \in \mathbb{R}$;

A3. $a+(b+c) = (a+b)+c \in \mathbb{R}$ if $a,b,c \in \mathbb{R}$;

A4. There exists $0 \in \mathbb{R}$ such that $a+0=a$ for all $a \in \mathbb{R}$;

A5. For any $a \in \mathbb{R}$, there is $b \in \mathbb{R}$ such that $a+b=0$;

M1. $a \cdot b \in \mathbb{R}$ if $a,b \in \mathbb{R}$;

M2. $a \cdot b = b \cdot a$ if $a,b \in \mathbb{R}$;

M3. $a \cdot (b \cdot c) = (a \cdot b) \cdot c \in \mathbb{R}$ if $a,b,c \in \mathbb{R}$;

M4. There exists $1 \in \mathbb{R} \setminus \{0\}$ such that $a \cdot 1 = a$ for all $a \in \mathbb{R}$;

M5. For any $a \in \mathbb{R} \setminus \{0\}$, there is $b \in \mathbb{R}$ such that $a \cdot b = 1$;

D. $a \cdot (b+c) = a \cdot b + a \cdot c$ if $a,b,c \in \mathbb{R}$.

1. (a) State the completeness of $\mathbb{R}$;
   (b) Using the axioms (and point out which axiom is used at each step), show that $(ab)^{-1} = a^{-1} \cdot b^{-1}$ for $a,b \in \mathbb{R} \setminus \{0\}$.

Pf: 1) a) Sps $S \subseteq \mathbb{R}$ is a nonempty subset that is bounded above, then $\sup S$ exists in $\mathbb{R}$.

b) Uniqueness of multiplicative inverse: Sps $a \in \mathbb{R} \setminus \{0\}$ and $b, c \in \mathbb{R}$ s.t. $a \cdot b = 1$, $a \cdot c = 1$.

$$
\begin{aligned}
b &= b \cdot 1 && (M4) \\
&= b \cdot (a \cdot c) && (\text{assumption}) \\
&= (b \cdot a) \cdot c && (M3) \\
&= (a \cdot b) \cdot c && (M2) \\
&= 1 \cdot c && (\text{assumption}) \\
&= c \cdot 1 && (M2) \\
&= c. && (M4)
\end{aligned}
$$

By uniqueness, it makes sense to talk about <u>the</u> multiplicative inverse of $a \in \mathbb{R}\setminus\{0\}$, which we will denote by a^{-1}.

Then, replacing a with ab in M5 above, and by uniqueness, we know that $(ab)^{-1}$ is the multiplicative inverse of ab. So we have

$$
\begin{aligned}
(ab)^{-1} &= (ab)^{-1}\cdot 1 && (M4)\\
&= (ab)^{-1}(aa^{-1}) && (M5)\\
&= (ab)^{-1}(aa^{-1})\cdot 1 && (M4)\\
&= (ab)^{-1}(aa^{-1})\cdot(bb^{-1}) && (M5)\\
&= (ab)^{-1}(a\cdot a^{-1}b)\cdot b^{-1} && (M3)\\
&= (ab)^{-1}(a\cdot b\cdot a^{-1})\cdot b^{-1} && (M2)\\
&= \cancel{(ab)^{-1}(a\cdot b)}\cdot a^{-1}\cdot b^{-1} && (M3)\\
&= 1\cdot a^{-1}\cdot b^{-1} && (M5)\\
&= a^{-1}\cdot b^{-1}\cdot 1 && (M2)\\
&= a^{-1}\cdot b^{-1} && (M4).
\end{aligned}
$$

2. (a) Let S be a non-empty subset of $\mathbb{R}$ *bounded from above*, show that $\sup(aS) = a \cdot \sup S$ if $a > 0$, *and where* $aS = \{a \cdot s : s \in S\}$.

   (b) Find $\sup S$ if $S = \{n^{-1} - m^{-1} : m, n \in \mathbb{N}\}$. Justify your answer.

Pf: a) S nonempty, bounded above, so by completeness axiom, $\sup S$ exists. (likewise need to check that aS is nonempty and bounded above to show $\sup(aS)$ exists).

Let $u = \sup S$. WTS $au = \sup(aS)$.

Since $u = \sup S$, $\forall s \in S$, $s \leq u$

$$\Downarrow a>0$$

$as \leq au$., so au is an u.b. of aS.

Sps v is an u.b. of aS. So $a \cdot s \leq v$ for all $s \in S$.

$$\Downarrow a>0$$

$s \leq \frac{v}{a}$ for all $s \in S$.

So $\frac{v}{a}$ is an u.b. of S. and we have

$$u \leq \frac{v}{a} \underset{a>0}{\Rightarrow} au \leq v;$$ as required. //

b) We'll show $\sup S = 1$: 1 is an u.b. of S: Since $n \in \mathbb{N}$, $0 < n^{-1} \leq 1$ and likewise $m^{-1} > 0$. So we obtain,

$$-m^{-1} < n^{-1} - m^{-1} \leq 1 - m^{-1} < 1. \checkmark$$

Now sps v is an u.b. of S with $v < 1$. Then $1 - v > 0$

By A.P. can find $m \in \mathbb{N}$ s.t. $\frac{1}{m} < 1 - v$.

Which gives $\underbrace{\frac{1}{1} - \frac{1}{m}}_{\in S} > v$, a contradiction. Hence $1 = \sup S$. //

3. We want to define x^r for $x > 1$ and $r \in \mathbb{Q}$ following the below procedure.

   (a) For any $w > 0$, show that for any $m \in \mathbb{N}$, there exists an unique $z \in \mathbb{R}$ such that $z > 0$ and $z^m = w$. You might use the fact that

$$(x+y)^n = \sum_{k=0}^{n} C_k^n x^{n-k} y^k, \quad \forall x, y \in \mathbb{R}.$$

   (b) Suppose $r = m/n = p/q \in \mathbb{Q}$ for some $m, n, p, q \in \mathbb{N}$. Show that

$$(x^m)^{1/n} = (x^p)^{1/q}$$

   where the quotient power is defined using (a). Thus, it makes sense to define $x^r := (x^m)^{1/n}$ where m/n is one of the representative of r.

   (c) Show that $x^{r_1+r_2} = x^{r_1} \cdot x^{r_2}$ for $r_1, r_2 \in \mathbb{Q}$ and $x > 1$;

   (d) (Bonus) Show that $x^r = \sup \mathcal{A}_r$ where $\mathcal{A}_r = \{x^p : p \le r, p \in \mathbb{Q}\}$. (Thus, we might define x^y for $y \in \mathbb{R}$, by $\sup \mathcal{A}_y$)

Pf: a) let $w>0$, $m \in \mathbb{N}$ be given. ~~Set~~ $S = \{ s \in \mathbb{R}$ s.t. $s^m < w\}$.

1st: WTS $\sup S$ ~~exists~~: Clearly $0^m = 0 \in S$. So S nonempty, bounded above: If $w<1$, then S bounded above by 1,
$w \geq 1$, then S bounded above by w.

So by completeness, $z := \sup S$ exists in $\mathbb{R}$.

2nd: $z > 0$: Since $w>0$, $\frac{w}{2} > 0$, and $\left(\frac{w}{2}\right)^m = \frac{w^m}{2^m} < w^m$
so $\frac{w}{2} \in S$, hence $z \geq \frac{w}{2} > 0$.

3rd $z^m = w$:

1st suppose $z^m < w$. Then want to find $n \in \mathbb{N}$ s.t.
$z + \frac{1}{n} \in S$, which is a contradiction b/c. $z + \frac{1}{n} > z$ and contradicts the fact that z is an u.b. of S.

$$\left(z + \frac{1}{n}\right)^m = \sum_{k=0}^{m} C_k^m \frac{z^{m-k}}{n^k} \qquad \left(n \geq 1, \text{ so } n^{-k} \leq n^{-1}\right)$$

$$\leq z^m + \frac{1}{n} \sum_{k=1}^{m} C_k^m z^{m-k}$$

Since $w - z^m > 0$, $z > 0 \Rightarrow z^{m-k} > 0$,

$\Rightarrow \sum_{k=1}^{m} C_k^m z^{m-k} > 0$, so we have

$$\frac{w - z^m}{\sum_{k=1}^{m} C_k^m z^{m-k}} > 0$$ and by A.P., can find $n \in \mathbb{N}$ s.t.

$$\frac{1}{n} < \frac{w - z^m}{\sum_{k=1}^{m} C_k^m z^{m-k}}$$

$$< \cancel{z^m} + \frac{w - \cancel{z^m}}{\cancel{\sum_{k=1}^{m} C_k^m z^{m-k}}} \left(\cancel{\sum_{k=1}^{m} C_k^m z^{m-k}} \right) = w.$$

So $z + \frac{1}{n} \in S$.

Sps $z^m > w$. WT find $n \in \mathbb{N}$ s.t. $z - \frac{1}{n}$ is an u.b. of S which contradicts the fact that z was l.u.b.

n TBD

$$\left(z - \frac{1}{n}\right)^m = z^m + \sum_{k=1}^{m} C_k^m \frac{z^{m-k}}{(-n)^k}$$

$$\geq z^m - \sum_{k=2l-1}^{m} C_k^m \frac{z^{m-k}}{n^k} \qquad l = 1, \dots \left\lfloor \frac{m+1}{2} \right\rfloor$$

$$\geq z^m - \frac{1}{n} \sum_{k=2l-1}^{m} C_k^m z^{m-k}$$

So similarly, by A.P. can find $n \in \mathbb{N}$ s.t.

$$\frac{1}{n} < \frac{z^m - w}{\sum_{k=2l-1}^{m} C_k^m z^{m-k}}$$

and substituting back in yields the desired contradiction.

So $z^m = w$.

Uniqueness: Sps $y \in \mathbb{R}$ s.t. $y > 0$, $y^m = w$. Then

$$0 = w - w = z^m - y^m$$

$$= (z-y)\underbrace{(z^{m-1} + z^{m-2}y + z^{m-3}y^2 + \cdots + z^2 y^{m-3} + z y^{m-2} + y^{m-1})}_{\text{by } z, y > 0 \text{, this factor } > 0}$$

$\Rightarrow z = y$. //

b) $r = \frac{m}{n} = \frac{p}{q}$. WTS $(x^m)^{\frac{1}{n}} = (x^p)^{\frac{1}{q}}$

by part (a), $(x^m)^{\frac{1}{n}}$ is the unique positive number z_1 s.t.

$$z_1^n = x^m.$$

Similarly $(x^p)^{\frac{1}{q}}$ is the unique positive number z_2 s.t.

$$z_2^q = x^p.$$

$\frac{m}{n} = \frac{p}{q} \implies mq = np$.

So we have $z_1^{nq} = x^{mq} = x^{np} = z_2^{nq}$.

So by uniqueness of part (a), we have that $z_1 = z_2$.

c) WTS $x^{r_1+r_2} = x^{r_1} \cdot x^{r_2}$ for $r_1, r_2 \in \mathbb{Q}$.

let $r_1 = \frac{m}{n}$, $r_2 = \frac{p}{q}$, for $m, n, p, q \in \mathbb{N}$. then $mq, pn \in \mathbb{N}$ and

$$\left(x^{r_1+r_2}\right)^{nq} = \left(x^{\frac{m}{n}+\frac{p}{q}}\right)^{nq} = x^{mq+pn} = x^{mq}x^{pn} = \left(x^{r_1}\right)^{nq} \cdot \left(x^{r_2}\right)^{nq}$$

$$= \left(x^{r_1} \cdot x^{r_2}\right)^{nq}$$

$$\left(x^{\frac{m}{n}+\frac{p}{q}}\right) = x^{\frac{mq+pn}{nq}}$$

By part a) $\exists! z_1 > 0$ s.t.

$$z_1^{nq} = x^{mq+pn}$$

So again by uniqueness from part (a),

$$x^{r_1+r_2} = x^{r_1} \cdot x^{r_2}.$$

d) $r \in \mathbb{Q}$. $A = \{x^p : p \le r,\ p \in \mathbb{Q}\}$.

1st show x^r is u.b. of A. Let $r = \frac{m}{n}$, $p = \frac{a}{b}$, where $m, n, a, b \in \mathbb{N}$.

So we have $rn = m$, $pb = a$. and also $p \le r$ gives $\frac{a}{b} \le \frac{m}{n}$.

$\Rightarrow an \le mb$.

$$(x^p)^{bn} = x^{na} \le x^{mb} = (x^r)^{nb}.$$

We can also show if $z^m \geq y^m$, then $z \geq y$ for $z, y > 0$:

$$\begin{aligned} 0 &\le z^m - y^m \\ &= (z-y)\underbrace{(z^{m-1} + z^{m-2}y + \cdots + zy^{m-2} + y^{m-1})}_{>0} \end{aligned}$$

$\Rightarrow z - y \geq 0 \Rightarrow z \geq y$.

This gives us $x^p \leq x^r$

l.u.b. : since $x^r \in A$. Sps for the sake of contradiction that we have an u.b. $v \in \mathbb{R}$ s.t. $v < x^r$. But this contradicts the fact that v is an u.b. of A, since $x^r \in A$.

Hence $x^r = \sup A$.